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Antosik–Mikusinski Matrix Convergence Theorem In Quantum Logics

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In this paper we establish the order topology type Antosik–Mikusinski infinite matrix convergence theorem in quantum logics. As application, we prove the Hahn–Schur summation theorem in quantum logics, too.

KEY WORDS: Antosik–Mikusinski theorem; infinite matrix; Order topology; quantum logics; Effect algebras.

1. EFFECT ALGEBRA AND ITS ELEMENTARY PROPERTIES

It is well known that the study of measure convergence theory on quantum logics is important for establishing the mathematical foundation of quantum mechanics. Many measure convergence theorems for measures defined on quantum logics and taking values in Abelian topological groups have been obtained (d'Andrea and deLucia, 1991; Habil, 1995; Mazario, 2001). Nevertheless, we are much more interested in those measure convergence theorems for measures which are defined on quantum logics and take values also in quantum logics. In this paper we present an elementary tool for studying such problems, that is, we prove an order topology type Antosik–Mikusinski infinite matrix convergence theorem on quantum logics. The classical Antosik–Mikusinski infinite matrix convergence theorem has very extensive applications in studying various topics in functional analysis and measure theory (Antosik and Swartz, 1985; Swartz, 1996). As application of the new Antosik–Mikusinski theorem we prove the Hahn–Schur summation theorem on quantum logics, too.

To model unsharp quantum logics, Foulis and Bennett (1994) introduced the following famous effect algebras:

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Let L be a set with two special elements $0, 1$, $\perp$ be a subset of $L \times L$, if $(a, b) \in \perp$, we denote $a \perp b$, and let $\oplus : \perp \to L$ be a binary operation. We say that the algebraic system $(L, \perp, \oplus, 0, 1)$ is an *effect algebra* if the following axioms hold:

(i) (Commutative law) If $a, b \in L$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
(ii) (Associative law) If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(iii) (Orthocomplementation law) For each $a \in L$ there exists a unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$.
(iv) (Zero-unit law) If $a \in L$ and $1 \perp a$, then $a = 0$.

Let $(L, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in L$ and $a \perp b$ we say that a and b be orthogonal. If $a \oplus b = 1$ we say that b is the orthocomplement of a, and write $b = a'$. It is clear that $1' = 0$, $(a')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in L$.

We also say that $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We may prove that $\leq$ is a partial order on L and satisfies that $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b' \leq a'$ and $a \leq b' \Leftrightarrow a \perp b$ for $a, b \in L$. If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b')'$. It will be denoted by $c = b \ominus a$. If $a \leq b$ but $a \neq b$, we write $a < b$.

The above showed that each effect algebra $(L, \perp, \oplus, 0, 1)$ has two binary operations $\oplus$ and $\ominus$.

If the partial order $\leq$ of effect algebra $(L, \perp, \oplus, 0, 1)$ defined as above is a lattice, then the effect algebra $(L, \perp, \oplus, 0, 1)$ is said to be a *lattice effect algebra*; if for all $a, b \in L$, $a \leq b$ or $b \leq a$, then $(L, \perp, \oplus, 0, 1)$ is said to be a *totally order effect algebra*; if for all $a, b \in L$, $a < b$, there exists $c \in L$ such that $a < c < b$, then $(L, \perp, \oplus, 0, 1)$ is said to be connected.

Let $F = \{a_i : 1 \leq i \leq n\}$ be a finite subset of L. If $a_1 \perp a_2$, $(a_1 \oplus a_2) \perp a_3, \ldots$ and $(a_1 \oplus a_2 \ldots \oplus a_{n-1}) \perp a_n$, we say that F is *orthogonal* and we define $\oplus F = a_1 \oplus a_2 \ldots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ (by the commutative and associative laws, this sum does not depend of any permutation of elements). Now, if A is an arbitrary subset of L and $\mathcal{F}(A)$ is the family of all finite subsets of A, we say that A is orthogonal if F is orthogonal for each $F \in \mathcal{F}(A)$. If A is orthogonal and the supremum $\bigvee\{\oplus F : F \in \mathcal{F}(A)\}$ exists, then $\oplus A = \bigvee\{\oplus F : F \in \mathcal{F}(A)\}$ is called the $\oplus$-sum of A.

An effect algebra is complete, if for each orthogonal subsets A of L, the $\oplus$-sum $\oplus A$ exists; if for each countable orthogonal subset B of L, the $\oplus$-sum $\oplus B$ exists, then we say that the effect algebra is *σ-complete*.

2. ORDER TOPOLOGY OF EFFECT ALGEBRAS

A partial order set $(\Lambda, \preceq)$ is said to be a *directed set*, if for all $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ such that $\alpha \preceq \gamma$, $\beta \preceq \gamma$.

If $(\Lambda, \preceq)$ is a directed set and for each $\alpha \in \Lambda$, $a_\alpha \in (L, \perp, \oplus, 0, 1)$, then $\{a_\alpha\}_{\alpha\in\Lambda}$ is said to be a net of $(L, \perp, \oplus, 0, 1)$.

Let $\{a_\alpha\}_{\alpha\in\Lambda}$ be a net of $(L, \perp, \oplus, 0, 1)$. Then we write $a_\alpha \uparrow$, when $\alpha \preceq \beta$, $a_\alpha \leq a_\beta$. Moreover, if a is the supremum of $\{a_\alpha : \alpha \in \Lambda\}$, i.e., $a = \vee\{a_\alpha : \alpha \in \Lambda\}$, then we write $a_\alpha \uparrow a$.

Similarly, we may write $a_\alpha \downarrow$ and $a_\alpha \downarrow a$.

If $\{u_\alpha\}_{\alpha\in\Lambda}$, $\{v_\alpha\}_{\alpha\in\Lambda}$ are two nets of $(L, \perp, \oplus, 0, 1)$, for $u \uparrow u_\alpha \leq v_\alpha \downarrow v$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$ and $v_\alpha \downarrow v$. We write $b \leq u_\alpha \uparrow u$ if $b \leq u_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$.

We say a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ is *order convergent* to a point a of L if there exists two nets $\{u_\alpha\}_{\alpha\in\Lambda}$ and $\{v_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.$$

Let $\mathcal{F} = \{F : F = \emptyset$ or $F \subseteq L$ and for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of F such that if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $a \in F\}$.

It is easy to prove that $\emptyset, L \in \mathcal{F}$ and if $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$, if $\{F_\mu\}_{\mu\in\Omega} \subseteq \mathcal{F}$, then $\bigcap_{\mu\in\Omega} F_\mu \in \mathcal{F}$. Thus, the family $\mathcal{F}$ of subsets of L define a topology τ_0^L on $(L, \perp, \oplus, 0, 1)$ such that $\mathcal{F}$ consists of all closed sets of this topology. The topology τ_0^L is called the *order topology* of $(L, \perp, \oplus, 0, 1)$ (Birkhoff, 1948).

We can prove that the order topology τ_0^L of $(L, \perp, \oplus, 0, 1)$ is the finest (strongest) topology on L such that for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$, if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $\{a_\alpha\}_{\alpha\in\Lambda}$ must be topology τ_0^L convergent to a. But the converse is not true.

When $(L, \perp, \oplus, 0, 1)$ is a lattice effect algebra, Riecanova (1999) proved the continuity of $\oplus$ and $\ominus$ with respect to the order topology as follows:

Theorem A. *If $(L, \perp, \oplus, 0, 1)$ is a lattice effect algebra, then a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ has*

(1) If $a_\alpha \leq b'$ for all $\alpha \in \Lambda$ and $\{a_\alpha\}_{\alpha\in\Lambda}$ convergent to a with respect to the order topology τ_0^L, then $\{a_\alpha \oplus b\}$ convergent to $a \oplus b$ with respect to the order topology τ_0^L.

(2) If $b \leq a_\alpha$ for all $\alpha \in \Lambda$ and $\{a_\alpha\}$ convergent to a with respect to the order topology τ_0^L, then $\{a_\alpha \ominus b\}$ convergent to $a \ominus b$ with respect to the order topology τ_0^L.

(3) If $a_\alpha \leq b$ for all $\alpha \in \Lambda$ and $\{a_\alpha\}$ convergent to a with respect to the order topology τ_0^L, then $\{b \ominus a_\alpha\}$ convergent to $b \ominus a$ with respect to the order topology τ_0^L.

From Theorem A we can prove the following important conclusion:

Theorem 1. *Let $(L, \perp, \oplus, 0, 1)$ be a totally order effect algebra. If $A = \{a_k\}_{k\in\mathbf{N}}$ is orthogonal $\oplus$-summable, then $\{a_n\}_{n\in\mathbf{N}}$ is order topology τ_0^L convergent to* 0.

In fact, since $(L, \perp, \oplus, 0, 1)$ is a totally order effect algebra, $\{[0, h) : h \in L\}$ is a neighbourhood basis of 0 of the order topology τ_0^L. Let $a = \oplus A = \bigvee\{\oplus_{k=1}^n a_k : n \in \mathbf{N}\}$, for each $h \in L$, $0 < h$ and note that $\{\oplus_{k=1}^n a_k\} \uparrow a$. Then it follows from Theorem A that there exists $n_0 \in \mathbf{N}$, such that for $n_0 \leq n$,

$$0 \leq a \ominus \left(\oplus_{k=1}^n a_k\right) < h.$$

So for $n_0 + 1 \leq n$ we have

$$a_n = \left(a \ominus \left(\oplus_{k=1}^{n-1} a_k\right)\right) \ominus \left(a \ominus \left(\oplus_{k=1}^{n} a_k\right)\right) \leq a \ominus \left(\oplus_{k=1}^{n-1} a_k\right) < h.$$

This shows that $\{a_n\}_{n\in\mathbf{N}}$ is order topology τ_0^L convergent to 0. This completes the proof.

The following lemmas and definition will be used in this paper.

Lemma 1. *If $(L, \perp, \oplus, 0, 1)$ is a σ-complete effect algebra, $\{a_i\}$ and $\{b_i\}$ two orthogonal $\oplus$-summable sequences of L and for each $i \in \mathbf{N}$, $b_i \leq a_i$. Then we have*

$$\vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n (a_i \ominus b_i)\right\} = \vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n a_i\right\} \ominus \vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n b_i\right\}.$$

Proof: Let $a_i \ominus b_i = c_i$. Then $\{c_i\}$ is an orthogonal sequences of L, by the σ-completeness of L, $\vee_{n\in\mathbf{N}}\{\oplus_{i=1}^n c_i\}$ exists. Note that $b_i \oplus c_i = a_i$, so we have

$$\vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n a_i\right\} = \vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n b_i\right\} \oplus \vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n c_i\right\}.$$

Thus, we have

$$\vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n (a_i \ominus b_i)\right\} = \vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n a_i\right\} \ominus \vee_{n\in\mathbf{N}}\left\{\oplus_{i=1}^n b_i\right\}.$$

The lemma is proved. □

Lemma 2. *If $(L, \perp, \oplus, 0, 1)$ is a σ-complete totally order connect effect algebra, then for each $h \in L$, $0 < h$, there exists an orthogonal $\oplus$-summable sequence $\{h_i\}$ of L such that $\vee_{n\in\mathbf{N}}\{\oplus_{i=1}^n h_i\} < h$.*

In fact, since $(L, \perp, \oplus, 0, 1)$ is a totally order connect effect algebra, so there exists $h_1, h_0 \in L$, such that $0 < h_1 < h_0 < h$. For $h_0 \ominus h_1$, there exists $h_2 \in L$ such that $0 < h_2 < h_0 \ominus h_1$. Similar, there exists $h_3 \in L$ such that $h_3 < (h_0 \ominus h_1) \ominus h_2$. Inductively, we can obtain an orthogonal sequence $\{h_i\}$ of L such that for each $n \in \mathbf{N}$, $\oplus_{i=1}^n h_i < h_0$. It following from the σ-completeness of L that $\vee_{n\in\mathbf{N}}\{\oplus_{i=1}^n h_i\}$ exists and $\vee_{i\in\mathbf{N}}\{\oplus_{i=1}^n h_i\} \leq h_0 < h$. This lemma is proved.

Definition 1. Let $(L, \perp, \oplus, 0, 1)$ be a totally order effect algebra. We say that the sequence $\{a_n\}_{n\in\mathbf{N}}$ of L is a τ_0^L-*Cauchy sequence*, if for each $h \in L$, $0 < h$, there exists $n_0 \in \mathbf{N}$ such that when $n_0 \le n$, $n_0 \le m$, if $a_n \le a_m$, then $a_m \ominus a_n < h$; if $a_m \le a_n$, then $a_n \ominus a_m < h$.

3. MAIN THEOREM AND ITS PROOF

We now prove the order topology type Antosik–Mikusinski infinite matrix convergent theorem on the σ-complete totally order connect effect algebras.

Theorem 2. *Let $(L, \perp, \oplus, 0, 1)$ be a σ-complete totally order connect effect algebra, $a_{ij} \in L$ for $i, j \in \mathbf{N}$. Suppose*

(I) $\{a_{ij}\}$ is order topology τ_0^L convergent to a_j for each $j \in \mathbf{N}$;
(II) For each $i \in \mathbf{N}$, $\{a_{ij}\}_{j\in\mathbf{N}}$ is an orthogonal sequence of L, and for each strictly increasing sequence of positive integers $\{m_j\}$, there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that the sequence $\{\oplus_j a_{in_j}\}_{i\in\mathbf{N}}$ is a τ_0^L-Cauchy sequence.

Then $\{a_{ij}\}$ is τ_0^L convergent to a_j uniformly for $j \in \mathbf{N}$. In particular, the diagonal $\{a_{ii}\}$ is τ_0^L convergent to 0.

Proof: If the conclusion fails, there exist $h \in L$, $0 < h$ and two strictly increasing sequences of positive integers $\{p_k\}$ and $\{q_k\}$ such that for all $k \in \mathbf{N}$, $h \le a_{q_k} \ominus a_{p_k q_k}$ when $a_{p_k q_k} \le a_{q_k}$; $h \le a_{p_k q_k} \ominus a_{q_k}$ when $a_{q_k} \le a_{p_k q_k}$. Without loss generality, we may assume that for all $k \in \mathbf{N}$, $a_{q_k} \le a_{p_k q_k}$, i.e.,

$$h \le a_{p_k q_k} \ominus a_{q_k}, k \in \mathbf{N} \tag{1}$$

By Lemma 2, we can choose $h_1, h_2 \in L$, $0 < h_1$, $0 < h_2$ such that $h_1 \oplus h_2 < h$. Note that for each $j \in \mathbf{N}$, $\{a_{p_i q_j}\}$ is τ_0^L convergent to a_{q_j}. Therefore, there exists a subsequence $\{m_i\}$ of $\{p_i\}$ such that for each $i \in \mathbf{N}$, if $a_{q_i} \le a_{m_i q_i}$, then $a_{m_i q_i} \ominus a_{q_i} < h_1$; if $a_{m_i q_i} \le a_{q_i}$, then $a_{q_i} \ominus a_{m_i q_i} < h_1$.

Note that when $a_{q_i} \le a_{m_i q_i}$, it follows from (1) that $a_{m_i q_i} \ominus a_{q_i} < h_1 < h \le a_{p_i q_i} \ominus a_{q_i}$, so $a_{m_i q_i} \le a_{p_i q_i}$; when $a_{m_i q_i} \le a_{q_i}$, it follows from (1) also that $a_{m_i q_i} \le a_{q_i} < a_{p_i q_i}$. Thus, for all $i \in \mathbf{N}$, $a_{m_i q_i} \le a_{p_i q_i}$.

On the other hand, if $a_{q_i} \le a_{m_i q_i}$, then

$$a_{p_i q_i} \ominus a_{q_i} = \left(a_{p_i q_i} \ominus a_{m_i q_i}\right) \oplus \left(a_{m_i q_i} \ominus a_{q_i}\right). \tag{2}$$

If $a_{m_i q_i} \le a_{q_i}$, then

$$a_{p_i q_i} \ominus a_{q_i} \le a_{p_i q_i} \ominus a_{m_i q_i} \tag{3}$$

Thus, if we can prove that for sufficient large i, $a_{p_iq_i} \ominus a_{m_iq_i} \le h_2$, then it follows from (2) and (3) that for sufficiently large i, $a_{p_iq_i} \ominus a_{q_i} < h$. This contradicts (1), which proves this theorem.

Now, let us consider the infinite matrix $(x_{ij})_{i,j\in\mathbf{N}}$, where $x_{ij} = a_{p_iq_j} \ominus a_{m_iq_j}$ if $a_{m_iq_j} \le a_{p_iq_j}$; $x_{ij} = a_{m_iq_j} \ominus a_{p_iq_j}$, if $a_{p_iq_j} \le a_{m_iq_j}$. It follows easily from the condition (I) and Theorem 1 that the matrix $(x_{ij})_{i,j\in\mathbf{N}}$ has the following properties:

(i) $\{x_{ij}\}_{i\in\mathbf{N}}$ is order topology τ_0^L convergent to 0 for each $j \in \mathbf{N}$;
(ii) $\{x_{ij}\}_{j\in\mathbf{N}}$ is order topology τ_0^L convergent to 0 for each $i \in \mathbf{N}$.

Now, we show that when i is sufficiently large, $x_{ii} \le h_2$.

If not, there exists an increasing sequence of positive integers $\{r_i\}$ such that for each $i \in \mathbf{N}$, $h_2 < x_{r_ir_i}$. Without loss of generality, we may assume that for all $i \in \mathbf{N}$,

$$h_2 < x_{ii}. \tag{4}$$

By Lemma 2, we can take an orthogonal $\oplus$-summable sequence $\{g_i\}$ of L such that $\vee_{n\in\mathbf{N}}\{\oplus_{i=1}^n g_i\} < h_2$; and for each g_i, we can take an orthogonal $\oplus$-summable sequence $\{g_{ij}\}$ of L such that $\vee_{n\in\mathbf{N}}\{\oplus_{j=1}^n g_{ij}\} < g_i$.

Let $l_1 = 1$. Then it follows from the properties of (i) and (ii) that we can find an index l_2 such that $x_{l_il_j} < g_{ij}$ for $i, j = 1, 2$, and $i \ne j$. By induction we can find an increasing sequence l_i such that

$$x_{l_il_j} < g_{ij}, i, j \in \mathbf{N}, \quad i \ne j. \tag{5}$$

It follows from the condition (II) that we can obtain a subsequence $\{s_i\}$ of $\{l_i\}$, without loss generality, we may also assume that the subsequence $\{s_i\}$ is just $\{l_i\}$, such that the sequence $\{\oplus_j a_{il_j}\}_{i\in\mathbf{N}}$ is a τ_0^L-Cauchy sequence. So the subsequence $\{\oplus_j a_{l_il_j}\}_{i\in\mathbf{N}}$ of $\{\oplus_j a_{il_j}\}_{i\in\mathbf{N}}$ is also a τ_0^L-Cauchy sequence.

Let $h_0 = h_2 \ominus \vee_{n\in\mathbf{N}} \{\oplus_{i=1}^n g_i\}$. Then there exists $i_0 \in \mathbf{N}$ such that when $i_0 \le i_1, i_0 \le i_2$, if $\oplus_j a_{l_{i_2}l_j} \le \oplus_j a_{l_{i_1}l_j}$, then $\oplus_j a_{l_{i_1}l_j} \ominus (\oplus_j a_{l_{i_2}l_j}) \le h_0$; if $\oplus_j a_{l_{i_1}l_j} \le \oplus_j a_{l_{i_2}l_j}$, then $\oplus_j a_{l_{i_2}l_j} \ominus (\oplus_j a_{l_{i_1}l_j}) \le h_0$.

Without loss generality, we may assume that $\oplus_j a_{l_{m_{i_0}}l_j} \le \oplus_j a_{l_{p_{i_0}}l_j}$, so

$$\oplus_j a_{l_{p_{i_0}}l_j} \ominus \left(\oplus_j a_{l_{m_{i_0}}l_j}\right) \le h_0. \tag{6}$$

Let $\triangle_1 = \{j : j \in \mathbf{N}, a_{l_{m_{i_0}}l_j} \le a_{l_{p_{i_0}}l_j}\}$, $\triangle_2 = \{j : j \in \mathbf{N}, a_{l_{p_{i_0}}l_j} < a_{l_{m_{i_0}}l_j}\}$. Then $\triangle_1 \cap \triangle_2 = \emptyset$ and $\triangle_1 \cup \triangle_2 = \mathbf{N}$. Furthermore, it follows from Lemma 1 that

$$\begin{aligned}&\vee_{j\in\triangle_1}\left(\oplus_j a_{l_{p_{i_0}}l_j} \ominus a_{l_{m_{i_0}}l_j}\right) \ominus \left(\vee_{j\in\triangle_2}\left(a_{l_{m_{i_0}}l_j} \ominus a_{l_{p_{i_0}}l_j}\right)\right)\\ &= \oplus_j a_{l_{p_{i_0}}l_j} \ominus \oplus_j a_{l_{m_{i_0}}l_j} \le h_0.\end{aligned}$$

Thus, it follows from the definitions of $\{g_i\}$ and h_0 that $a_{l_{p_{i_0}}l_{p_{i_0}}} \ominus a_{l_{m_{i_0}}l_{m_{i_0}}} < h_2$. This contradicts (4), which proves this theorem. □

Now, we apply Theorem 2 to prove the Hahn-Schur Theorem on Effect Algebras.

Theorem 3. *Let* $(L, \perp, \oplus, 0, 1)$ *be a* σ*-complete totally order connect effect algebra, for each* $i \in \mathbf{N}$*,* $\{a_{ij}\}_{j \in \mathbf{N}}$ *be an orthogonal sequence of* L*. If for each subset* $\triangle$ *of* $\mathbf{N}$*, the* $\oplus$*-sum sequence* $\{\oplus_{j\in\triangle}{}^{a_{ij}}\}_{i\in\mathbf{N}}$ *is order topology* τ_0^L *convergent, then* $\{a_{ij}\}_{j\in\mathbf{N}}$ *are uniformly* $\oplus$*-summable with respect to* $i \in \mathbf{N}$*.*

Proof: If not, there exist a $h \in L$, $0 < h$, and a sequence of finite sets $\{\triangle_k\}$ of $\mathbf{N}$ such that $\max \triangle_k < \min \triangle_{k+1}$ for all $k \in \mathbf{N}$, a strictly increasing positive integers sequence $\{i_k\}$ such that

$$h \leq \oplus_{j\in\triangle_k} a_{i_k j}, \quad k \in \mathbf{N}. \tag{7}$$

Using Theorem 2, we can easily prove that (7) is impossible. The theorem is proved. □

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